

Sharp convergence rates of time discretization for stochastic time-fractional PDEs subject to additive space-time white noise

Max Gunzburger*

Buyang Li†

Jilu Wang‡

Abstract

The stochastic time-fractional equation $\partial_t \psi - \Delta \partial_t^{1-\alpha} \psi = f + \dot{W}$ with space-time white noise \dot{W} is discretized in time by a backward-Euler convolution quadrature for which the sharp-order error estimate

$$\mathbb{E} \|\psi(\cdot, t_n) - \psi_n\|_{L^2(\mathcal{O})}^2 = O(\tau^{1-\alpha d/2})$$

is established for $\alpha \in (0, 2/d)$, where d denotes the spatial dimension, ψ_n the approximate solution at the n^{th} time step, and \mathbb{E} the expectation operator. In particular, the result indicates optimal convergence rates of numerical solutions for both stochastic subdiffusion and diffusion-wave problems in one spatial dimension. Numerical examples are presented to illustrate the theoretical analysis.

Keywords: stochastic partial differential equation, time-fractional derivative, space-time white noise, error estimates

1 Introduction

We are interested in the convergence of numerical methods for solving the stochastic time-fractional partial differential equation (PDE) problem

$$\begin{cases} \partial_t \psi(x, t) - \Delta \partial_t^{1-\alpha} \psi(x, t) = f(x, t) + \dot{W}(x, t) & (x, t) \in \mathcal{O} \times \mathbb{R}_+ \\ \psi(x, t) = 0 & (x, t) \in \partial\mathcal{O} \times \mathbb{R}_+ \\ \psi(x, 0) = \psi_0(x) & x \in \mathcal{O}, \end{cases} \quad (1.1)$$

where $\mathcal{O} \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, denotes a bounded region with Lipschitz boundary $\partial\mathcal{O}$, $f(x, t)$ a given deterministic source function, $\psi_0(x)$ given deterministic initial condition, and $\dot{W}(x, t)$ a space-time white noise, i.e., the time derivative of a cylindrical Wiener process in $L^2(\mathcal{O})$. The underlying probability sample space for the stochastic noise is denoted by Ω . The operator $\Delta : D(\Delta) \rightarrow L^2(\mathcal{O})$ denotes the Laplacian, defined on the domain

$$D(\Delta) = \{\phi \in H_0^1(\mathcal{O}) : \Delta \phi \in L^2(\mathcal{O})\},$$

*Department of Scientific Computing, Florida State University, Tallahassee, FL 32306, USA. (gunzburg@fsu.edu)

†Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong. (buyang.li@polyu.edu.hk)

‡Department of Scientific Computing, Florida State University, Tallahassee, FL 32306, USA. (jwang13@fsu.edu)

and $\partial_t^{1-\alpha}\psi$ denotes the left-sided Caputo fractional time derivative of order $1 - \alpha \in (-1, 1)$, defined by (c.f. [10, pp. 91])

$$\partial_t^{1-\alpha}\psi(x, t) := \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{\partial\psi(x, s)}{\partial s} ds & \text{if } \alpha \in (0, 1], \\ \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \psi(x, s) ds & \text{if } \alpha \in (1, 2), \end{cases} \quad (1.2)$$

where $\Gamma(s) := \int_0^\infty t^{s-1} e^{-t} dt$ denotes the Euler gamma function.

Problem (1.1) arises naturally by considering the heat equation in a material with thermal memory, subject to stochastic noise [3, 11, 16]. For the model (1.1), both the fractional time derivative and the stochastic process forcing result in solution having low regularity. Hence, the numerical approximation of such problems and the corresponding numerical analysis are very challenging. By defining $\partial_t^\alpha\psi(x, t) := \partial_t^{\alpha-1}\partial_t\psi(x, t)$ for $\alpha \in (1, 2)$ and using the identity

$$\partial_t^{\alpha-1}\partial_t^{1-\alpha}\psi(x, t) = \begin{cases} \psi(x, t) - \psi(x, 0) & \text{if } \alpha \in (0, 1), \\ \psi(x, t) & \text{if } \alpha \in (1, 2), \end{cases} \quad (1.3)$$

applying $\partial_t^{\alpha-1}$ to (1.1) yields another formulation of (1.1):

$$\partial_t^\alpha\psi(x, t) - \Delta\psi(x, t) = \begin{cases} \partial_t^{\alpha-1}(f(x, t) + \dot{W}(x, t)) - \Delta\psi(x, 0) & \text{if } \alpha \in (0, 1), \\ f(x, t) + \dot{W}(x, t) - \Delta\psi(x, 0) & \text{if } \alpha = 1, \\ \partial_t^{\alpha-1}(f(x, t) + \dot{W}(x, t)) & \text{if } \alpha \in (1, 2), \end{cases} \quad (1.4)$$

where the case $\alpha = 1$ can be verified directly from (1.1). For the sake of clarity, we focus on only one of the equivalent problems, namely (1.1).

The solution of (1.1) can be decomposed into the solution of the deterministic problem

$$\begin{cases} \partial_t v(x, t) - \Delta\partial_t^{1-\alpha}v(x, t) = f(x, t) & (x, t) \in \mathcal{O} \times \mathbb{R}_+ \\ v(x, t) = 0 & (x, t) \in \partial\mathcal{O} \times \mathbb{R}_+ \\ v(x, 0) = \psi_0(x) & x \in \mathcal{O} \end{cases} \quad (1.5)$$

plus the solution of the stochastic problem

$$\begin{cases} \partial_t u(x, t) - \Delta\partial_t^{1-\alpha}u(x, t) = \dot{W}(x, t) & (x, t) \in \mathcal{O} \times \mathbb{R}_+ \\ u(x, t) = 0 & (x, t) \in \partial\mathcal{O} \times \mathbb{R}_+ \\ u(x, 0) = 0 & x \in \mathcal{O}, \end{cases} \quad (1.6)$$

The stability and convergence of numerical solutions of (1.5) have been widely studied [2, 4, 14, 15, 17]. For example, if f is smooth in time then numerical methods of up to order 2 are available for approximating the solution of (1.5) and its equivalent formulations [8, 9, 14, 15]. In particular, the convolution quadrature generated by the backward Euler method yields a first-order convergence rate for solving (1.5).

In this work, we focus on numerical approximation of the stochastic time-fractional PDE (1.6) with additive space-time white noise based on the convolution quadrature generated by the backward Euler method. In the case $\alpha \in (1, 2)$ and $d = 1$, rigorous error estimates for numerical solutions of (1.6) are carried out in [11] for the case of additive Gaussian noise in the general Q -Wiener process setting. For a space-time white noise, an almost optimal-order convergence rate for the time discretization error

$$\mathbb{E}\|u(\cdot, t_n) - u_n\|_{L^2(\mathcal{O})}^2 = O(\tau^{1-\alpha/2-\epsilon})$$

is proved [11, Remark 4.7, with $\rho = \alpha$] for arbitrarily small ϵ , where $u(\cdot, t_n)$ and u_n denote the mild solution and numerical solution of (1.6) at time t_n , respectively. We are not aware of any

rigorous numerical analyses in the case $\alpha \in (0, 1)$. In the special case $\alpha = 1$, error estimates for time discretization of the stochastic PDE (1.6) are proved in [7] and [6, 18] for Rothe's method and the backward Euler method, respectively. In particular, the sharp convergence rate $O(\tau^{1/4})$ in time is proved for the expectation of the L^2 norm time discretization error [1].

The aim of this paper is to prove, for general d -dimensional domains, the sharp-order convergence rate estimate

$$\mathbb{E}\|u(\cdot, t_n) - u_n\|_{L^2(\mathcal{O})}^2 = O(\tau^{1-\alpha d/2}) \quad \alpha \in (0, 2/d), \quad d \in \{1, 2, 3\} \quad (1.7)$$

for time discretization of the stochastic PDE (1.6). This estimate is achieved via a more delicate analysis of the resolvent operator by using its Laplace transform representation. Our result covers both subdiffusion and diffusion-wave cases in one-dimensional spatial domains and, for the subdiffusion case, multi-dimensional domains.

The rest of the paper is organized as follows. In Section 2, we present the backward-Euler convolution quadrature scheme we use to determine approximate solutions of the stochastic time-fractional PDE (1.6) and then state our main theoretical results. In Section 3, we derive an integral representation of the numerical solution for which we prove sharp convergence rate results for the approximate solution. Numerical results are given in Section 4 to illustrate the theoretical analyses.

Throughout this paper, we denote by C , with/without a subscript, a generic constant independent of n and τ which could be different at different occurrences.

2 The main results

In this section, we describe the time discretization scheme we use for determining approximate solutions of the stochastic time-fractional PDE (1.1) and state our main results about the convergence rate of the numerical solutions.

2.1 Mild solution of the stochastic PDE

Let $\phi_j(x)$, $j = 1, 2, \dots$, denote the L^2 -norm normalized eigenfunctions of the Laplacian operator $-\Delta$ corresponding to the eigenvalues λ_j , $j = 1, 2, \dots$, arranged in nondecreasing order. The cylindrical Wiener process on $L^2(\mathcal{O})$ can be represented as (cf. [5, Proposition 4.7, with $Q = I$ and U_1 denoting some negative-order Sobolev space])

$$W(x, t) = \sum_{j=1}^{\infty} \phi_j(x) W_j(t) \quad (2.1)$$

with independent one-dimensional Wiener processes $W_j(t)$, $j = 1, 2, \dots$.

In the case $\psi_0 = 0$, the solution of the deterministic problem (1.5) can be expressed by (via Laplace transform, cf. [14, (3.11) and line 4 of page 12])

$$v(\cdot, t) = \int_0^t E(t-s) f(\cdot, s) ds, \quad (2.2)$$

where the operator $E(t) : L^2(\mathcal{O}) \rightarrow L^2(\mathcal{O})$ is given by

$$E(t)\phi := \frac{1}{2\pi i} \int_{\Gamma_{\theta, \kappa}} e^{zt} z^{\alpha-1} (z^\alpha - \Delta)^{-1} \phi dz \quad \forall \phi \in L^2(\mathcal{O}), \quad (2.3)$$

with integration over a contour $\Gamma_{\theta, \kappa}$ on the complex plane,

$$\Gamma_{\theta, \kappa} = \{z \in \mathbb{C} : |z| = \kappa, |\arg z| \leq \theta\} \cup \{z \in \mathbb{C} : z = \rho e^{\pm i\theta}, \rho \geq \kappa\}$$

$$=: \Gamma_{\theta, \kappa}^{\kappa} + \Gamma_{\theta, \kappa}^{\theta}. \quad (2.4)$$

The angle θ above can be any angle such that $\pi/2 < \theta < \min(\pi, \pi/\alpha)$ so that, for all z to the right of $\Gamma_{\theta, \kappa}$ in the complex plane, $z^\alpha \in \Sigma_{\alpha\theta} := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| \leq \alpha\theta\}$ with $\alpha\theta < \pi$.

Correspondingly, the mild solution of (1.6) is define as (cf. [16] and [11, Proposition 2.7])

$$u(\cdot, t) = \int_0^t E(t-s) dW(\cdot, s) \quad (2.5)$$

$$= \sum_{j=1}^{\infty} \int_0^t E(t-s) \phi_j dW_j(s). \quad (2.6)$$

This mild solution is well defined in $C([0, T]; L^2(\Omega; L^2(\mathcal{O})))$; see the Appendix A.

2.2 Convolution quadrature

Let $\{t_n = n\tau\}_{n=0}^N$ denote a uniform partition of the interval $[0, T]$ with a time step size $\tau = T/N$, and let $u^n = u(x, t_n)$. Under the zero initial condition, the Caputo fractional time derivative $\partial_t^{1-\alpha} u(x, t_n)$ can be discretized by the backward-Euler convolution quadrature $\bar{\partial}_\tau^{1-\alpha} u_n$ (also known as Grünwald-Letnikov approximation, cf. [4]) defined by

$$\bar{\partial}_\tau^{1-\alpha} u_n = \frac{1}{\tau^{1-\alpha}} \sum_{j=0}^n b_{n-j} u_j, \quad n = 0, 1, 2, \dots, N, \quad (2.7)$$

where b_j , $j = 0, 1, 2, \dots, N$, are the coefficients in the power series expansion

$$(1 - \zeta)^{1-\alpha} = \sum_{j=0}^{\infty} b_j \zeta^j. \quad (2.8)$$

Here, $1 - \zeta$ is the characteristic function of the backward-Euler method and we set

$$\delta_\tau(\zeta) = \frac{1 - \zeta}{\tau} \quad \text{for } \zeta \in \mathbb{C} \setminus [1, \infty). \quad (2.9)$$

For any sequence $\{v_n\}_{n=0}^{\infty} \in \ell^2(L^2(\mathcal{O}))$, we denote the generating function of the sequence by

$$\tilde{v}(\zeta) = \sum_{n=0}^{\infty} v_n \zeta^n \quad \text{for } \zeta \in \mathbb{D} \quad (2.10)$$

that is an $L^2(\mathcal{O})$ -valued analytic function in the unit disk \mathbb{D} and the limit

$$\tilde{v}(e^{i\theta}) = \lim_{r \rightarrow 1_-} \tilde{v}(re^{i\theta})$$

exists in $L^2(0, 2\pi; L^2(\Omega))$. Then, we have

$$\begin{aligned} \sum_{n=0}^{\infty} (\bar{\partial}_\tau^{1-\alpha} v_n) \zeta^n &= \sum_{n=0}^{\infty} \frac{1}{\tau^{1-\alpha}} \sum_{j=0}^n b_{n-j} v_j \zeta^n \\ &= (\delta_\tau(\zeta))^{1-\alpha} \sum_{j=0}^{\infty} v_j \zeta^j = (\delta_\tau(\zeta))^{1-\alpha} \tilde{v}(\zeta). \end{aligned} \quad (2.11)$$

2.3 Time-stepping scheme and main theorem

With the notations introduced in the last subsection, we discretize the fractional-order derivative $\partial_t^{1-\alpha}$ in (1.6) by using convolution quadrature in time to obtain

$$\frac{u_n - u_{n-1}}{\tau} - \Delta \bar{\partial}_\tau^{1-\alpha} u_n = \frac{W(\cdot, t_n) - W(\cdot, t_{n-1})}{\tau}. \quad (2.12)$$

Equivalently, u_n can be expressed as

$$\begin{aligned} u_n &= (1 - \tau^\alpha b_0 \Delta)^{-1} u_{n-1} + \tau^\alpha \sum_{j=0}^{n-1} b_{n-j} \Delta (1 - \tau^\alpha b_0 \Delta)^{-1} u_j \\ &\quad + (1 - \tau^\alpha b_0 \Delta)^{-1} (W(\cdot, t_n) - W(\cdot, t_{n-1})) \\ &= (1 - \tau^\alpha b_0 \Delta)^{-1} u_{n-1} + \tau^\alpha \sum_{j=0}^{n-1} b_{n-j} \Delta (1 - \tau^\alpha b_0 \Delta)^{-1} u_j \\ &\quad + \sum_{j=1}^{\infty} (W_j(t_n) - W_j(t_{n-1})) (1 + \tau^\alpha b_0 \lambda_j)^{-1} \phi_j. \end{aligned} \quad (2.13)$$

The main result of this paper is the following theorem.

Theorem 2.1. *Let $\alpha \in (0, 2/d)$ with $d \in \{1, 2, 3\}$. Then, for each $n = 1, 2, \dots, N$, the numerical solution u_n given by (2.12) is well defined in $L^2(\Omega; L^2(\mathcal{O}))$ and converges to the mild solution $u(\cdot, t_n)$ with sharp order of convergence, i.e., we have*

$$\max_{1 \leq n \leq N} \mathbb{E} \|u(\cdot, t_n) - u_n\|_{L^2(\mathcal{O})}^2 \leq C \tau^{1-\alpha d/2}, \quad (2.14)$$

where \mathbb{E} denotes the expectation operator and the constant C is independent of T .

3 Proof of Theorem 2.1

3.1 The numerical solution in $L^2(\Omega; L^2(\mathcal{O}))$

In this subsection, we show that the numerical solution is well defined in $L^2(\Omega; L^2(\mathcal{O}))$. To this end, we use the following estimate for the eigenvalues of the Laplacian operator. For the simplicity of notations, we denote by (\cdot, \cdot) and $\|\cdot\|$ the inner product and norm of $L^2(\mathcal{O})$, respectively.

Lemma 3.1 ([12, 13]). *Let \mathcal{O} denote a bounded domain in \mathbb{R}^d , $d \in \{1, 2, 3\}$. Suppose λ_j denotes the j^{th} eigenvalue of the Dirichlet boundary problem for the Laplacian operator $-\Delta$ in \mathcal{O} . With $|\mathcal{O}|$ denoting the volume of \mathcal{O} , we have that*

$$\lambda_j \geq \frac{C_d d}{d+2} j^{2/d} |\mathcal{O}|^{-2/d} \quad (3.1)$$

for all $j \geq 1$, where $C_d = (2\pi)^2 B_d^{-2/d}$ and B_d denotes the volume of the unit d -ball.

Clearly, if $u_j \in L^2(\Omega; L^2(\mathcal{O}))$ for $j = 0, \dots, n-1$, then

$$(1 - \tau^\alpha b_0 \Delta)^{-1} u_{n-1} + \tau^\alpha \sum_{j=0}^{n-1} b_{n-j} \Delta (1 - \tau^\alpha b_0 \Delta)^{-1} u_j \in L^2(\Omega; L^2(\mathcal{O})). \quad (3.2)$$

In view of (2.13), we only need to prove

$$\sum_{j=1}^{\infty} (W_j(t_n) - W_j(t_{n-1})) (1 + \tau^\alpha b_0 \lambda_j)^{-1} \phi_j \in L^2(\Omega; L^2(\mathcal{O})). \quad (3.3)$$

In fact, we have

$$\begin{aligned} & \mathbb{E} \left\| \sum_{j=\ell}^{\ell+m} (W_j(t_n) - W_j(t_{n-1})) (1 + \tau^\alpha b_0 \lambda_j)^{-1} \phi_j \right\|^2 \\ &= \mathbb{E} \sum_{j=\ell}^{\ell+m} |W_j(t_n) - W_j(t_{n-1})|^2 (1 + \tau^\alpha b_0 \lambda_j)^{-2} = \sum_{j=\ell}^{\ell+m} \tau (1 + \tau^\alpha b_0 \lambda_j)^{-2} \\ &\leq C b_0^{-2} \tau^{1-2\alpha} \sum_{j=\ell}^{\ell+m} j^{-4/d} \rightarrow 0 \quad \text{as } \ell \rightarrow \infty. \end{aligned}$$

Hence, for a fixed time step τ ,

$$\sum_{j=1}^{\ell} (W_j(t_n) - W_j(t_{n-1})) (1 + \tau^\alpha b_0 \lambda_j)^{-1} \phi_j, \quad \ell = 1, 2, \dots$$

is a Cauchy sequence in $L^2(\Omega; L^2(\mathcal{O}))$. Consequently, (3.3) is proved. In view of (2.13) and (3.2)-(3.3), the numerical solution u_n is well defined in $L^2(\Omega; L^2(\mathcal{O}))$.

3.2 A technical lemma

To prove the error estimate in Theorem 2.1, we need the following technical lemma.

Lemma 3.2.

$$\sum_{j=1}^{\infty} \left(\frac{r^\alpha}{r^\alpha + \lambda_j} \right)^2 \leq C r^{\alpha d/2} \quad \forall r > 0, \quad (3.4)$$

$$\left| \frac{1}{z + \lambda_j} \right| \leq \frac{C_\varphi}{|z| + \lambda_j} \quad \forall z \in \Sigma_\varphi \text{ with } \varphi \in (0, \pi). \quad (3.5)$$

Proof. Clearly, Lemma 3.1 implies $\lambda_j \geq C j^{2/d}$.

First, if $0 < r < 1$,

$$\sum_{j=1}^{\infty} \left(\frac{r^\alpha}{r^\alpha + \lambda_j} \right)^2 \leq \sum_{j=1}^{\infty} \frac{r^{2\alpha}}{C j^{4/d}} \leq C r^{2\alpha} \leq C r^{\alpha d/2}, \quad (d/2 \leq 2 \text{ for } d = 1, 2, 3). \quad (3.6)$$

Second, if $r \geq 1$, by setting $M = [r^{\alpha d/2}] \geq 1$ to be the largest integer that does not exceed $r^{\alpha d/2}$, we have

$$\begin{aligned} \sum_{j=1}^{\infty} \left(\frac{r^\alpha}{r^\alpha + \lambda_j} \right)^2 &\leq \sum_{j=1}^{\infty} \left(\frac{r^\alpha}{r^\alpha + C j^{2/d}} \right)^2 \\ &= \sum_{j=1}^{M+1} \left(\frac{r^\alpha}{r^\alpha + C j^{2/d}} \right)^2 + \sum_{j=M+2}^{\infty} \left(\frac{r^\alpha}{r^\alpha + C j^{2/d}} \right)^2 =: I_1 + I_2. \end{aligned}$$

It is easy to see that $I_1 \leq M + 1 \leq 2M \leq 2r^{\alpha d/2}$ and

$$I_2 \leq \int_{r^{\alpha d/2}}^{\infty} \left(\frac{r^\alpha}{r^\alpha + Cs^{2/d}} \right)^2 ds = C^{-d/2} r^{\alpha d/2} \int_{C^{d/2}}^{\infty} \left(\frac{1}{1 + \xi^{2/d}} \right)^2 d\xi \leq Cr^{\alpha d/2},$$

where the equality follows by changing the variable $s = C^{-d/2} r^{\alpha d/2} \xi$. This proves (3.4) in the case $r \geq 1$.

Finally, for the point $\xi = -\lambda_j + 0i$ in the complex plane, we have $|\xi| = \lambda_j$ and $|z - \xi| = |z + \lambda_j|$. By looking at the triangle with three vertices z , 0 , and ξ with interior angles ω_z , ω_0 , and ω_ξ at the three vertices, respectively, we have

$$\frac{|z - \xi|}{\sin(\omega_0)} = \frac{|z|}{\sin(\omega_\xi)} = \frac{\lambda_j}{\sin(\omega_z)}.$$

If $\omega_0 \geq \pi/2$, then $|z - \xi|$ would be the length of the longest side of the triangle, i.e.,

$$|z - \xi| \geq |z| \quad \text{and} \quad |z - \xi| \geq \lambda_j$$

which immediately implies

$$|z - \xi| \geq \frac{1}{2}(|z| + \lambda_j).$$

If $\omega_0 \leq \pi/2$, then the angle condition $|\arg(z)| < \varphi$ implies $\omega_0 > \pi - \varphi$. Hence, we have

$$|z - \xi| = \frac{|z| \sin(\omega_0)}{\sin(\omega_\xi)} \geq |z| \sin(\varphi) \quad \text{and} \quad |z - \xi| = \frac{\lambda_j \sin(\omega_0)}{\sin(\omega_z)} \geq \lambda_j \sin(\varphi)$$

which immediately implies

$$|z - \xi| \geq \frac{\sin(\varphi)}{2}(|z| + \lambda_j).$$

In either case, we have (3.5). This completes the proof of Lemma 3.2. \square

3.3 Solution representations

In this subsection, we derive a representation of the semidiscrete solution u_n by means of the discrete analogue of the Laplace transform and generating function.

Let $\Gamma_{\theta, \kappa}^{(\tau)}$ denote the truncated piece of the contour $\Gamma_{\theta, \kappa}$ defined by

$$\Gamma_{\theta, \kappa}^{(\tau)} := \{z \in \Gamma_{\theta, \kappa} : |\operatorname{Im}(z)| \leq \pi/\tau\}. \quad (3.7)$$

For $\rho \in (0, 1)$, let $\Gamma_\rho^{(\tau)}$ denote the segment of a vertical line defined by

$$\Gamma_\rho^{(\tau)} := \{z = -\ln(\rho)/\tau + iy : y \in \mathbb{R} \text{ and } |y| \leq \pi/\tau\}. \quad (3.8)$$

The following technical lemma will be used in this and next subsections.

Lemma 3.3. *Let $\alpha \in (0, 2/d)$ and $\theta \in (\frac{\pi}{2}, \operatorname{arccot}(-\frac{2}{\pi}))$ be given, and let $\rho \in (0, 1)$ be fixed, with $\delta_\tau(\zeta)$ defined in (2.9). Then, both $\delta(e^{-z\tau})$ and $(\delta(e^{-z\tau})^\alpha - \Delta)^{-1}$ are analytic with respect to z in the region enclosed by*

$$\Gamma_\rho^{(\tau)}, \quad \Gamma_{\theta, \kappa}^{(\tau)}, \quad \text{and the two lines } \mathbb{R} \pm i\pi/\tau \quad \text{whenever} \quad 0 < \kappa \leq \min(1/T, -\ln(\rho)/\tau).$$

Furthermore, we have the following estimates:

$$\delta_\tau(e^{-z\tau}) \in \Sigma_\theta \quad \forall z \in \Gamma_{\theta, \kappa}^{(\tau)} \quad (3.9)$$

$$C_0|z| \leq |\delta_\tau(e^{-\tau z})| \leq C_1|z| \quad \forall z \in \Gamma_{\theta, \kappa}^{(\tau)} \quad (3.10)$$

$$|\delta_\tau(e^{-\tau z}) - z| \leq C\tau|z|^2 \quad \forall z \in \Gamma_{\theta, \kappa}^{(\tau)} \quad (3.11)$$

$$|\delta_\tau(e^{-\tau z})^\alpha - z^\alpha| \leq C\tau|z|^{\alpha+1} \quad \forall z \in \Gamma_{\theta, \kappa}^{(\tau)}, \quad (3.12)$$

where the constants C_0 , C_1 , and C are independent of τ and $\kappa \in (0, \min(\frac{1}{T}, -\frac{\ln(\rho)}{\tau}))$.

Proof. Clearly, (3.9) is a consequence of the following two inequalities:

$$0 \leq \arg\left(\frac{1 - e^{-z\tau}}{\tau}\right) \leq \arg(z) \quad \text{if } 0 \leq \arg(z) \leq \theta, \quad (3.13)$$

$$-\arg(z) \leq \arg\left(\frac{1 - e^{-z\tau}}{\tau}\right) \leq 0 \quad \text{if } -\theta \leq \arg(z) \leq 0, \quad (3.14)$$

which can be proved in the following way when $\frac{\pi}{2} \leq \theta \leq \operatorname{arccot}(-\frac{2}{\pi})$.

If $\arg(z) = \varphi \in [0, \theta]$ and $0 \leq \operatorname{Im}(z) \leq \pi/\tau$ (thus $0 \leq \tau|z|\sin(\varphi) \leq \pi$), then it is easy to see that $\arg(\frac{1 - e^{-\tau z}}{\tau}) \geq 0$ and

$$\begin{aligned} \cot\left(\arg\left(\frac{1 - e^{-\tau z}}{\tau}\right)\right) &= \frac{1 - e^{-\tau|z|\cos(\varphi)} \cos(\tau|z|\sin(\varphi))}{e^{-\tau|z|\cos(\varphi)} \sin(\tau|z|\sin(\varphi))} \\ &= \frac{e^{\tau|z|\cos(\varphi)} - \cos(\tau|z|\sin(\varphi))}{\sin(\tau|z|\sin(\varphi))} \\ &\geq \frac{1 + \tau|z|\cos(\varphi) - \cos(\tau|z|\sin(\varphi))}{\sin(\tau|z|\sin(\varphi))} \quad (\text{Taylor's expansion}) \\ &= \frac{1 + \omega \cot(\varphi) - \cos(\omega)}{\sin(\omega)} \quad (\text{set } \omega = \tau|z|\sin(\varphi) \in [0, \pi]). \end{aligned}$$

We shall prove $\cot(\arg(\frac{1 - e^{-\tau z}}{\tau})) \geq \cot(\varphi)$ so that $0 \leq \arg(\frac{1 - e^{-\tau z}}{\tau}) \leq \varphi = \arg(z)$. To this end, we consider the function

$$g(\omega) := 1 + \omega \cot(\varphi) - \cos(\omega) - \sin(\omega) \cot(\varphi), \quad \omega \in [0, \pi].$$

By considering the sign of the derivative $g'(\omega)$, it is easy to see that the function $g(\omega)$ achieves its minimum value at one of the two end points $\omega = 0$ and $\omega = \pi$, with

$$g(0) = 0 \quad \text{and} \quad g(\pi) = 2 + \pi \cot(\varphi).$$

If $\frac{\pi}{2} \leq \theta \leq \operatorname{arccot}(-\frac{2}{\pi})$, we then have $g(\pi) \geq 0$. Consequently, $g(\omega) \geq 0$ for all $\omega \in [0, \pi]$ and $\cot(\arg(\frac{1 - e^{-\tau z}}{\tau})) \geq \cot(\varphi)$ which implies

$$0 \leq \arg\left(\frac{1 - e^{-\tau z}}{\tau}\right) \leq \varphi = \arg(z).$$

This proves (3.13). The inequality (3.14) can be proved in the same way. This completes the proof of (3.9) which further implies that $\delta(e^{-z\tau})$ and $(\delta(e^{-z\tau})^\alpha - \Delta)^{-1}$ are analytic with respect to z in the region enclosed by

$$\Gamma_\rho^{(\tau)}, \Gamma_{\theta, \kappa}^{(\tau)} \quad \text{and the two lines } \mathbb{R} \pm i\pi/\tau \quad \text{whenever } 0 < \kappa \leq -\ln(\rho)/\tau.$$

The estimates (3.10)-(3.12) are simple consequences of Taylor's theorem. \square

To derive the representation of the numerical solution u_n , we introduce some notations. Let $\bar{\partial}_\tau W$ be defined by

$$\bar{\partial}_\tau W(\cdot, t_0) := 0 \quad (3.15)$$

$$\bar{\partial}_\tau W(\cdot, t) := \frac{W(\cdot, t_n) - W(\cdot, t_{n-1})}{\tau} \quad \text{for } t \in (t_{n-1}, t_n], \quad n = 1, 2, \dots, N \quad (3.16)$$

$$\bar{\partial}_\tau W(\cdot, t) := 0 \quad \text{for } t > t_N, \quad (3.17)$$

where we have set $\bar{\partial}_\tau W(\cdot, t) = 0$ for $t > t_N$; this does not affect the value of u_n , $n = 1, 2, \dots, N$, upon solving (2.12). Similarly, we define

$$\bar{\partial}_\tau W_j(t_0) := 0 \quad (3.18)$$

$$\bar{\partial}_\tau W_j(t) := \frac{W_j(t_n) - W_j(t_{n-1})}{\tau} \quad \text{for } t \in (t_{n-1}, t_n], \quad n = 1, 2, \dots, N \quad (3.19)$$

$$\bar{\partial}_\tau W_j(t) := 0 \quad \text{for } t > t_N. \quad (3.20)$$

With these definitions, there are only a finite number of nonzero terms in the sequence $\bar{\partial}_\tau W(\cdot, t_n)$, $n = 0, 1, 2, \dots$. Consequently, the generating function

$$\widetilde{\bar{\partial}_\tau W}(\cdot, \zeta) = \sum_{n=0}^{\infty} \bar{\partial}_\tau W(\cdot, t_n) \zeta^n$$

is well defined (polynomial in ζ). Then, we have the following result.

Proposition 3.1. *For the time-stepping scheme (2.12), the semidiscrete solution u_n can be represented by*

$$u_n = \int_0^{t_n} E_\tau(t_n - s) \bar{\partial}_\tau W(\cdot, s) ds \quad (3.21)$$

$$= \sum_{j=1}^{\infty} \int_0^{t_n} E_\tau(t_n - s) \phi_j \bar{\partial}_\tau W_j(s) ds, \quad (3.22)$$

where the operator $E_\tau(\cdot)$ is given by

$$E_\tau(t)\phi := \frac{1}{2\pi i} \int_{\Gamma_{\theta, \kappa}^{(\tau)}} e^{zt} \frac{z\tau}{e^{z\tau} - 1} \delta_\tau(e^{-z\tau})^{\alpha-1} (\delta_\tau(e^{-z\tau})^\alpha - \Delta)^{-1} \phi dz \quad \forall \phi \in L^2(\mathcal{O}) \quad (3.23)$$

with integration over the truncated contour $\Gamma_{\theta, \kappa}^{(\tau)}$ defined in (3.7), oriented with increasing imaginary parts, with the parameters κ and θ satisfying the conditions of Lemma 3.3.

Proof. In view of definition (2.10) and the identity (2.11), multiplying (2.12) by ζ^n and summing up the results over $n = 0, 1, 2, \dots$ yield

$$\delta_\tau(\zeta) \tilde{u}(\zeta) - \delta_\tau(\zeta)^{1-\alpha} \Delta \tilde{u}(\zeta) = \widetilde{\bar{\partial}_\tau W}(\cdot, \zeta). \quad (3.24)$$

Then,

$$\tilde{u}(\zeta) = \delta_\tau(\zeta)^{\alpha-1} (\delta_\tau(\zeta)^\alpha - \Delta)^{-1} \widetilde{\bar{\partial}_\tau W}(\cdot, \zeta). \quad (3.25)$$

The function $\tilde{u}(\zeta)$ defined in (3.25) is analytic with respect to ζ in a neighborhood of the origin. By Cauchy's integral formula, it implies that for $\rho \in (0, 1)$

$$u_n = \frac{1}{2\pi i} \int_{|\zeta|=\rho} \zeta^{-n-1} \tilde{u}(\zeta) d\zeta = \frac{\tau}{2\pi i} \int_{\Gamma_\rho^{(\tau)}} e^{zt_n} \tilde{u}(e^{-z\tau}) dz,$$

where the second equality is obtained by the change of variables $\zeta = e^{-z\tau}$, with the contour $\Gamma_\rho^{(\tau)}$ defined in (3.8).

From Lemma 3.3, we see that both $\delta(e^{-z\tau})$ and $(\delta(e^{-z\tau})^\alpha - \Delta)^{-1}$ are analytic with respect to z in the region $\Sigma \subset \mathbb{C}$ enclosed by $\Gamma_\rho^{(\tau)}$, $\Gamma_{\theta,\kappa}^{(\tau)}$, and the two lines $\mathbb{R} \pm i\pi/\tau$. Thus, $e^{zt_n}\tilde{u}(e^{-z\tau})$ is analytic with respect to $z \in \Sigma$. Because the values of $e^{zt_n}\tilde{u}(e^{-z\tau})$ on the two lines $\mathbb{R} \pm i\pi/\tau$ coincide, it follows that (by applying Cauchy's integral formula)

$$\begin{aligned}
u_n &= \frac{\tau}{2\pi i} \int_{\Gamma_\rho^{(\tau)}} e^{zt_n} \tilde{u}(e^{-z\tau}) dz \\
&= \frac{\tau}{2\pi i} \int_{\Gamma_{\theta,\kappa}^{(\tau)}} e^{zt_n} \tilde{u}(e^{-z\tau}) dz + \frac{\tau}{2\pi i} \int_{\mathbb{R} + \frac{i\pi}{\tau}} e^{zt_n} \tilde{u}(e^{-z\tau}) dz \\
&\quad - \frac{\tau}{2\pi i} \int_{\mathbb{R} - \frac{i\pi}{\tau}} e^{zt_n} \tilde{u}(e^{-z\tau}) dz \\
&= \frac{\tau}{2\pi i} \int_{\Gamma_{\theta,\kappa}^{(\tau)}} e^{zt_n} \tilde{u}(e^{-z\tau}) dz \\
&= \frac{\tau}{2\pi i} \int_{\Gamma_{\theta,\kappa}^{(\tau)}} e^{zt_n} \delta_\tau(e^{-z\tau})^{\alpha-1} (\delta_\tau(e^{-z\tau})^\alpha - \Delta)^{-1} \widehat{\bar{\partial}_\tau W}(\cdot, e^{-z\tau}) dz \\
&= \frac{\tau}{2\pi i} \int_{\Gamma_{\theta,\kappa}^{(\tau)}} e^{zt_n} \delta_\tau(e^{-z\tau})^{\alpha-1} (\delta_\tau(e^{-z\tau})^\alpha - \Delta)^{-1} \frac{z}{e^{z\tau} - 1} \widehat{\bar{\partial}_\tau W}(\cdot, z) dz, \tag{3.26}
\end{aligned}$$

where we have substituted (3.25) into the above equality and used the following (straightforward to check) identity in the last step:

$$\widehat{\bar{\partial}_\tau W}(\cdot, e^{-z\tau}) = \frac{z}{e^{z\tau} - 1} \widehat{\bar{\partial}_\tau W}(\cdot, z)$$

with $\widehat{\bar{\partial}_\tau W}$ denoting the Laplace transform (in time) of the piecewise constant function $\bar{\partial}_\tau W$.

Through the Laplace transform rule

$$\mathcal{L}^{-1}(\widehat{f\hat{g}})(t) = \int_0^t \mathcal{L}^{-1}(\widehat{f})(t-s) \mathcal{L}^{-1}(\widehat{g})(s) ds, \tag{3.27}$$

one can derive (3.21) from (3.26). The proof of Proposition 3.1 is complete. \square

3.4 Error estimate

In this subsection, we derive an error estimate for the numerical scheme (2.12). The following lemma is concerned with the difference between the kernels of (2.3) and (3.23). It will be used in the proof of Theorem 2.1.

Lemma 3.4. *Let $\alpha \in (0, 2/d)$ be given and let $\delta_\tau(\zeta)$ be defined as in (2.9) with the parameters κ and θ satisfying the conditions of Lemma 3.3. Then, we have*

$$\left| z^{\alpha-1} (z^\alpha + \lambda_j)^{-1} - \frac{z\tau}{e^{z\tau} - 1} \delta_\tau(e^{-\tau z})^{\alpha-1} (\delta_\tau(e^{-\tau z})^\alpha + \lambda_j)^{-1} \right| \leq \frac{C\tau|z|^\alpha}{|z|^\alpha + \lambda_j}, \quad \forall z \in \Gamma_{\theta,\kappa}^{(\tau)}.$$

Proof. By the triangle inequality and Lemma 3.3, we have

$$\begin{aligned}
&\left| z^{\alpha-1} (z^\alpha + \lambda_j)^{-1} - \frac{z\tau}{e^{z\tau} - 1} \delta_\tau(e^{-\tau z})^{\alpha-1} (\delta_\tau(e^{-\tau z})^\alpha + \lambda_j)^{-1} \right| \\
&\leq \left| \frac{e^{z\tau} - 1 - z\tau}{e^{z\tau} - 1} \right| |z^{\alpha-1} (z^\alpha + \lambda_j)^{-1}|
\end{aligned}$$

$$\begin{aligned}
& + \left| \frac{z\tau}{e^{z\tau} - 1} \right| |z|^{\alpha-1} |(z^\alpha + \lambda_j)^{-1} - (\delta_\tau(e^{-\tau z})^\alpha + \lambda_j)^{-1}| \\
& + \left| \frac{z\tau}{e^{z\tau} - 1} \right| |z^{\alpha-1} - \delta_\tau(e^{-\tau z})^{\alpha-1}| |(\delta_\tau(e^{-\tau z})^\alpha + \lambda_j)^{-1}| =: \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{J}_1 & \leq C|z\tau||z^{\alpha-1}(z^\alpha + \lambda_j)^{-1}| \quad (\text{using the Taylor expansion of } \left| \frac{e^{z\tau} - 1 - z\tau}{e^{z\tau} - 1} \right|) \\
\mathcal{J}_2 & \leq C|z|^{\alpha-1}|z^\alpha - \delta_\tau(e^{-\tau z})^\alpha| |(z^\alpha + \lambda_j)^{-1}(\delta_\tau(e^{-\tau z})^\alpha + \lambda_j)^{-1}| \\
& \leq C\tau|z|^{2\alpha}|(z^\alpha + \lambda_j)^{-1}(\delta_\tau(e^{-\tau z})^\alpha + \lambda_j)^{-1}| \quad (\text{here we use (3.12)}) \\
& \leq C\tau|z|^{2\alpha}(|z|^\alpha + \lambda_j)^{-1}(|\delta_\tau(e^{-\tau z})|^\alpha + \lambda_j)^{-1} \\
& \leq \frac{C\tau|z|^\alpha}{|z|^\alpha + \lambda_j}.
\end{aligned}$$

The last inequality is due to Lemma 3.2 together with the angle condition $\arg(z^\alpha) \leq \alpha\theta < \pi$ and $\arg(\delta_\tau(e^{-\tau z})^\alpha) \leq \alpha\theta < \pi$ (cf. Lemma 3.3). Furthermore, we have

$$\begin{aligned}
\mathcal{J}_3 & \leq C|z^{\alpha-1} - \delta_\tau(e^{-\tau z})^{\alpha-1}| |(\delta_\tau(e^{-\tau z})^\alpha + \lambda_j)^{-1}| \\
& \leq C \left(|z^\alpha - \delta_\tau(e^{-\tau z})^\alpha| |z|^{-1} + |z^{-1} - \delta_\tau(e^{-\tau z})^{-1}| |\delta_\tau(e^{-\tau z})|^\alpha \right) |(\delta_\tau(e^{-\tau z})^\alpha + \lambda_j)^{-1}| \\
& \leq \left(C\tau|z|^{1+\alpha}|z|^{-1} + C\tau|z|^2|z|^{-1}|\delta_\tau(e^{-\tau z})|^{\alpha-1} \right) |(\delta_\tau(e^{-\tau z})^\alpha + \lambda_j)^{-1}| \\
& \leq \frac{C\tau|z|^\alpha}{|z|^\alpha + \lambda_j} \quad (\text{here we use Lemma 3.2 and Lemma 3.3}).
\end{aligned}$$

The proof of Lemma 3.4 is complete. \square

Now, we start to prove Theorem 2.1. From (2.6) and (2.3) we see that the mild solution admits the decomposition

$$u(\cdot, t) = \sum_{j=1}^{\infty} \phi_j \int_0^t F_j^{(\tau)}(t-s) dW_j(s) + \sum_{j=1}^{\infty} \phi_j \int_0^t H_j^{(\tau)}(t-s) dW_j(s) \quad (3.28)$$

with

$$F_j^{(\tau)}(t) := \frac{1}{2\pi i} \int_{\Gamma_{\theta, \kappa}^{(\tau)}} e^{zt} z^{\alpha-1} (z^\alpha + \lambda_j)^{-1} dz \quad (3.29)$$

$$H_j^{(\tau)}(t) := \frac{1}{2\pi i} \int_{\Gamma_{\theta, \kappa} \setminus \Gamma_{\theta, \kappa}^{(\tau)}} e^{zt} z^{\alpha-1} (z^\alpha + \lambda_j)^{-1} dz. \quad (3.30)$$

Also, (3.21) and (3.23) imply

$$u_n = \sum_{j=1}^{\infty} \phi_j \int_0^{t_n} E_j^{(\tau)}(t_n - s) \bar{\partial}_\tau W_j(s) ds \quad (3.31)$$

with

$$E_j^{(\tau)}(t) := \frac{1}{2\pi i} \int_{\Gamma_{\theta, \kappa}^{(\tau)}} e^{zt} \frac{z\tau}{e^{z\tau} - 1} \delta_\tau(e^{-z\tau})^{\alpha-1} (\delta_\tau(e^{-z\tau})^\alpha + \lambda_j)^{-1} dz. \quad (3.32)$$

Comparing (3.28) and (3.31) yields

$$u(\cdot, t_n) - u_n = \sum_{j=1}^{\infty} \phi_j \int_0^{t_n} (F_j^{(\tau)}(t_n - s) - E_j^{(\tau)}(t - s)) dW_j(s)$$

$$\begin{aligned}
& + \sum_{j=1}^{\infty} \phi_j \int_0^{t_n} E_j^{(\tau)}(t_n - s) \left(dW_j(s) - \bar{\partial}_\tau W_j(s) ds \right) \\
& + \sum_{j=1}^{\infty} \phi_j \int_0^{t_n} H_j^{(\tau)}(t_n - s) dW_j(s) \\
& =: \mathcal{E}_\tau(t_n) + \mathcal{G}_\tau(t_n) + \mathcal{H}_\tau(t_n).
\end{aligned} \tag{3.33}$$

In the following, we separately estimate $\mathcal{H}_\tau(t_n)$, $\mathcal{E}_\tau(t_n)$, and $\mathcal{G}_\tau(t_n)$.

First, we estimate $\mathcal{H}_\tau(t_n)$. By choosing a number $\beta \in (\alpha d/2, 1)$ and using Lemma 3.2, we have

$$\begin{aligned}
\mathbb{E} \|\mathcal{H}_\tau(t_n)\|^2 &= \int_0^{t_n} \sum_{j=1}^{\infty} |H_j^{(\tau)}(t_n - s)|^2 ds = \int_0^{t_n} \sum_{j=1}^{\infty} |H_j^{(\tau)}(s)|^2 ds \\
&= \int_0^{t_n} \sum_{j=1}^{\infty} \left| \frac{1}{2\pi i} \int_{\Gamma_{\theta, \kappa} \setminus \Gamma_{\theta, \kappa}^{(\tau)}} e^{zs} z^{\alpha-1} (z^\alpha + \lambda_j)^{-1} dz \right|^2 ds \\
&\leq \int_0^{t_n} \sum_{j=1}^{\infty} \left| \frac{1}{2\pi i} \int_{\Gamma_{\theta, \kappa} \setminus \Gamma_{\theta, \kappa}^{(\tau)}} \frac{z^\alpha}{z^\alpha + \lambda_j} \frac{e^{zs}}{z} dz \right|^2 ds \\
&\leq C \int_0^{t_n} \sum_{j=1}^{\infty} \left(\int_{\Gamma_{\theta, \kappa} \setminus \Gamma_{\theta, \kappa}^{(\tau)}} \frac{|dz|}{|z|^{2-\beta}} \right) \left(\int_{\Gamma_{\theta, \kappa} \setminus \Gamma_{\theta, \kappa}^{(\tau)}} \left| \frac{z^\alpha}{z^\alpha + \lambda_j} \right|^2 \frac{|e^{zs}|^2}{|z|^\beta} |dz| \right) ds \\
&\leq C \int_0^{t_n} \sum_{j=1}^{\infty} \left(\int_{1/\tau}^{\infty} \frac{dr}{r^{2-\beta}} \right) \left(\int_{1/\tau}^{\infty} \left| \frac{r^\alpha}{r^\alpha + \lambda_j} \right|^2 \frac{e^{(2s \cos \theta)r}}{r^\beta} dr \right) ds \\
&\leq C \tau^{1-\beta} \int_0^{t_n} \int_{1/\tau}^{\infty} \sum_{j=1}^{\infty} \left(\frac{r^\alpha}{r^\alpha + \lambda_j} \right)^2 \frac{e^{(2s \cos \theta)r}}{r^\beta} dr ds \\
&\leq C \tau^{1-\beta} \int_0^{t_n} \int_{1/\tau}^{\infty} r^{\alpha d/2 - \beta} e^{(2s \cos \theta)r} dr ds \\
&\leq C \tau^{1-\beta} \int_{1/\tau}^{\infty} r^{\alpha d/2 - \beta - 1} (1 - e^{(2t_n \cos \theta)r}) dr \\
&\leq C \tau^{1-\beta} \tau^{\beta - \alpha d/2} \\
&\leq C \tau^{1 - \alpha d/2}.
\end{aligned} \tag{3.34}$$

Next, we estimate $\mathcal{E}_\tau(t_n)$. To this end, we apply Lemma 3.4 and obtain

$$\begin{aligned}
& |F_j^{(\tau)}(s) - E_j^{(\tau)}(s)|^2 \\
&= \left| \frac{1}{2\pi i} \int_{\Gamma_{\theta, \kappa}^{(\tau)}} e^{zs} \left(\frac{z^{\alpha-1}}{z^\alpha + \lambda_j} - \frac{z\tau}{e^{z\tau} - 1} \frac{\delta_\tau(e^{-\tau z})^{\alpha-1}}{\delta_\tau(e^{-\tau z})^\alpha + \lambda_j} \right) dz \right|^2 \\
&\leq \left(\int_{\Gamma_{\theta, \kappa}^{(\tau)}} |e^{zs}| \frac{C\tau |z|^\alpha}{|z|^\alpha + \lambda_j} |dz| \right)^2 \\
&\leq C\tau^2 \left(\int_{\Gamma_{\theta, \kappa}^{(\tau)}} |dz| \right) \int_{\Gamma_{\theta, \kappa}^{(\tau)}} \left(\frac{|z|^\alpha}{|z|^\alpha + \lambda_j} \right)^2 |e^{zs}|^2 |dz| \\
&\leq C\tau \int_{\Gamma_{\theta, \kappa}^{(\tau)}} \left(\frac{|z|^\alpha}{|z|^\alpha + \lambda_j} \right)^2 |e^{zs}|^2 |dz|.
\end{aligned} \tag{3.35}$$

By using the expression of $\mathcal{E}_\tau(t_n)$, we have

$$\begin{aligned}
\mathbb{E} \|\mathcal{E}_\tau(t_n)\|^2 &= \sum_{j=1}^{\infty} \mathbb{E} \left| \int_0^{t_n} (F_j^{(\tau)}(t_n - s) - E_j^{(\tau)}(t_n - s)) dW_j(s) \right|^2 \\
&= \sum_{j=1}^{\infty} \int_0^{t_n} |F_j^{(\tau)}(t_n - s) - E_j^{(\tau)}(t_n - s)|^2 ds \\
&= \sum_{j=1}^{\infty} \int_0^{t_n} |F_j^{(\tau)}(s) - E_j^{(\tau)}(s)|^2 ds \\
&\leq C\tau \int_0^{t_n} \int_{\Gamma_{\theta, \kappa}^{(\tau)}} \sum_{j=1}^{\infty} \left(\frac{|z|^\alpha}{|z|^\alpha + \lambda_j} \right)^2 |e^{zs}|^2 |dz| ds \\
&\leq C\tau \int_0^{t_n} \int_{\Gamma_{\theta, \kappa}^{(\tau)}} |z|^{\alpha d/2} |e^{zs}|^2 |dz| ds,
\end{aligned} \tag{3.36}$$

where the last inequality follows from Lemma 3.2. Since $t_n \geq \tau$ and

$$\Gamma_{\theta, \kappa}^{(\tau)} = \{z \in \mathbb{C} : z = re^{\pm i\theta}, r \geq \kappa, r|\sin(\theta)| \leq \pi/\tau\} \cup \{z \in \mathbb{C} : |z| = \kappa, |\arg z| \leq \theta\},$$

by choosing $\kappa \leq \frac{2}{t_n |\sin(\theta)|}$, we have

$$\begin{aligned}
\mathbb{E} \|\mathcal{E}_\tau(t_n)\|^2 &\leq C\tau \int_0^{t_n} \int_{\kappa}^{\frac{\pi}{\tau |\sin(\theta)|}} r^{\alpha d/2} e^{(2s \cos \theta)r} dr ds \\
&\quad + C\tau \int_0^{t_n} \int_{-\theta}^{\theta} \kappa^{\alpha d/2+1} e^{2s \cos(\psi)\kappa} d\psi ds \\
&\leq C\tau \int_{\kappa}^{\frac{\pi}{\tau |\sin(\theta)|}} r^{\alpha d/2-1} (1 - e^{(2t \cos \theta)r}) dr + C\tau \int_0^{t_n} \int_{-\theta}^{\theta} \kappa^{\alpha d/2+1} e^{2s \kappa} d\psi ds \\
&\leq C\tau^{1-\alpha d/2} + C\tau \kappa^{\alpha d/2} (e^{2t_n \kappa} - 1) \\
&\leq C\tau^{1-\alpha d/2}.
\end{aligned} \tag{3.37}$$

Finally, we estimate $\mathcal{G}_\tau(t_n)$. Because $\bar{\partial}_\tau W_j(t_n) = \frac{1}{\tau} \int_{t_{n-1}}^{t_n} dW_j(s)$, we obtain

$$\begin{aligned}
\mathcal{G}_\tau(t_n) &= \sum_{j=1}^{\infty} \phi_j \int_0^{t_n} E_j^{(\tau)}(t_n - s) (dW_j(s) - \bar{\partial}_\tau W_j(s) ds) \\
&= \sum_{j=1}^{\infty} \phi_j \sum_{i=1}^n \left(\int_{t_{i-1}}^{t_i} E_j^{(\tau)}(t_n - s) dW_j(s) - \int_{t_{i-1}}^{t_i} E_j^{(\tau)}(t_n - \xi) \bar{\partial}_\tau W_j(t_i) d\xi \right) \\
&= \sum_{j=1}^{\infty} \phi_j \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left(\frac{1}{\tau} \int_{t_{i-1}}^{t_i} (E_j^{(\tau)}(t_n - s) - E_j^{(\tau)}(t_n - \xi)) d\xi \right) dW_j(s).
\end{aligned} \tag{3.38}$$

Then,

$$\begin{aligned}
\mathbb{E} \|\mathcal{G}_\tau(t_n)\|^2 &= \sum_{j=1}^{\infty} \mathbb{E} \left| \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left(\frac{1}{\tau} \int_{t_{i-1}}^{t_i} (E_j^{(\tau)}(t_n - s) - E_j^{(\tau)}(t_n - \xi)) d\xi \right) dW_j(s) \right|^2 \\
&= \sum_{j=1}^{\infty} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left| \frac{1}{\tau} \int_{t_{i-1}}^{t_i} (E_j^{(\tau)}(t_n - s) - E_j^{(\tau)}(t_n - \xi)) d\xi \right|^2 ds.
\end{aligned} \tag{3.39}$$

By using the expression (3.32), for $|s - \xi| \leq \tau$ we have

$$\begin{aligned}
& \left| E_j^{(\tau)}(t_n - s) - E_j^{(\tau)}(t_n - \xi) \right|^2 \\
&= \left| \frac{1}{2\pi i} \int_{\Gamma_{\theta, \kappa}^{(\tau)}} e^{z(t_n - s)} (1 - e^{z(s - \xi)}) \delta_\tau(e^{-z\tau})^{\alpha-1} (\delta_\tau(e^{-z\tau})^\alpha + \lambda_j)^{-1} \frac{z\tau}{e^{z\tau} - 1} dz \right|^2 \\
&\leq C \left(\int_{\Gamma_{\theta, \kappa}^{(\tau)}} |dz| \right) \left(\int_{\Gamma_{\theta, \kappa}^{(\tau)}} |e^{z(t_n - s)}|^2 |1 - e^{z(s - \xi)}|^2 \left| \frac{\delta_\tau(e^{-z\tau})^{\alpha-1}}{\delta_\tau(e^{-z\tau})^\alpha + \lambda_j} \right|^2 \left| \frac{z\tau}{e^{z\tau} - 1} \right| |dz| \right) \\
&\leq C\tau^{-1} \int_{\Gamma_{\theta, \kappa}^{(\tau)}} |e^{z(t_n - s)}|^2 \tau^2 |z|^2 \left| \frac{\delta_\tau(e^{-z\tau})^{\alpha-1}}{\delta_\tau(e^{-z\tau})^\alpha + \lambda_j} \right|^2 |dz| \\
&\leq C\tau \int_{\Gamma_{\theta, \kappa}^{(\tau)}} |e^{z(t_n - s)}|^2 |z|^2 \left(\frac{|z|^{\alpha-1}}{|z|^\alpha + \lambda_j} \right)^2 |dz| \quad (\text{here we use Lemma 3.2}).
\end{aligned} \tag{3.40}$$

Substituting the last inequality into (3.39) yields

$$\begin{aligned}
\mathbb{E} \|\mathcal{G}_\tau(t_n)\|^2 &\leq \sum_{j=1}^{\infty} C\tau \int_0^{t_n} \int_{\Gamma_{\theta, \kappa}^{(\tau)}} |e^{z(t_n - s)}|^2 |z|^2 \left(\frac{|z|^{\alpha-1}}{|z|^\alpha + \lambda_j} \right)^2 |dz| ds \\
&\leq C\tau \int_0^{t_n} \int_{\Gamma_{\theta, \kappa}^{(\tau)}} |e^{z(t_n - s)}|^2 |z|^2 \sum_{j=1}^{\infty} \left(\frac{|z|^{\alpha-1}}{|z|^\alpha + \lambda_j} \right)^2 |dz| ds \\
&\leq C\tau \int_0^{t_n} \int_{\Gamma_{\theta, \kappa}^{(\tau)}} |e^{z(t_n - s)}|^2 |z|^{\alpha d/2} |dz| ds \\
&= C\tau \int_0^{t_n} \int_{\Gamma_{\theta, \kappa}^{(\tau)}} |e^{zs}|^2 |z|^{\alpha d/2} |dz| ds \quad (\text{here we use a change of variable}) \\
&\leq C\tau^{1-\alpha d/2},
\end{aligned} \tag{3.41}$$

where the last inequality can be estimated in the same way as (3.36).

Substituting (3.34), (3.37), and (3.41) into (3.33) yields

$$\mathbb{E} \|u(\cdot, t_n) - u_n\|^2 \leq C\tau^{1-\alpha d/2}. \tag{3.42}$$

The proof of Theorem 2.1 is complete.

4 Numerical examples

In this section, we present three numerical examples to illustrate the theoretical analyses.

Example 1. We first consider the one-dimensional stochastic time-fractional equation

$$\partial_t u(x, t) - \partial_x^2 \partial_t^{1-\alpha} u(x, t) = f(x, t) + \varepsilon \dot{W}(x, t) \tag{4.1}$$

for $0 \leq x \leq 1$, $0 < t \leq 1$, with homogenous Dirichlet boundary condition and 0 initial condition. In the above equation,

$$f(x, t) = 2tx^2(1-x)^2 - \frac{2t^{1+\alpha}}{\Gamma(2+\alpha)}(2-12x+12x^2),$$

ε is a given constant, and W the cylindrical Wiener process. In the absence of white noise, the exact solution would be $u_d(x, t) = t^2 x^2 (1 - x)^2$.

We discretize the problem (4.1) in time by using the scheme (2.12) and, in space, by continuous piecewise linear finite element method. Here, $h = 1/M$ denotes the spatial mesh size and $U^n(x)$ the numerical solution of the fully discrete scheme. We take $\tau = h = 2^{-5}$ and $\varepsilon = 0.1$. For each computation, $I = 1000$ independent realizations are performed with different Wiener processes. For each realization ω_i , $i = 1, \dots, I$, we generate M independent Brownian motions $W_j(t)$, $j = 1, \dots, M$. In Figure 4.1(left), we present the exact solution u_d of the deterministic problem, the mean value of numerical solutions for (4.1), and the standard deviation, respectively, at $t_n = 1$. Moreover, the numerical approximations $U^n(x, \omega_i)$, $i = 1, 2, 3$ of $u(x, t_n, \omega_i)$, with three independent realizations, are given in Figure 4.1(right) at $t_n = 1$. The numerical simulations in Figure 4.1 are performed by taking $\alpha = 0.5$. Similar results are shown in Figure 4.2 for $\alpha = 1.3$.

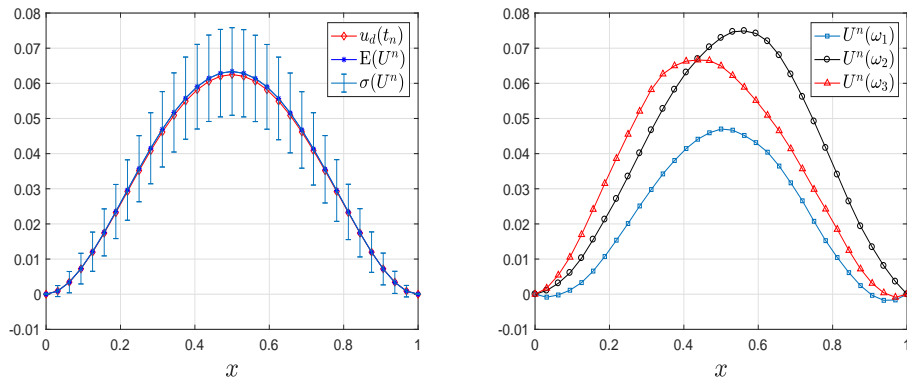


Figure 1: Numerical approximations for $u(x, t)$ with $\alpha = 0.5$

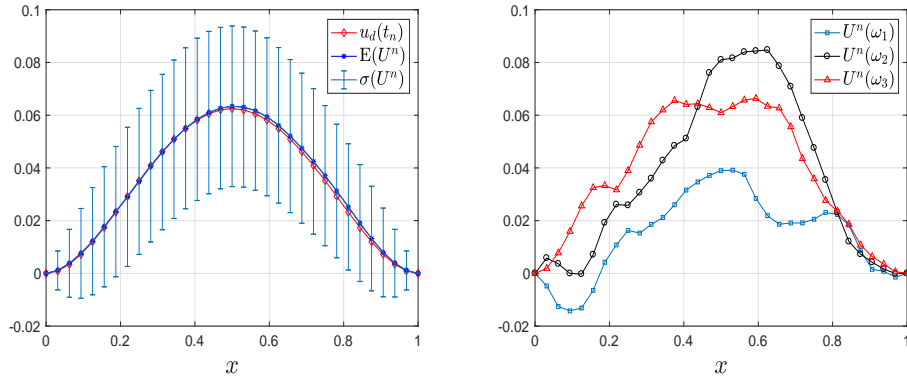


Figure 2: Numerical approximations for $u(x, t)$ with $\alpha = 1.3$

Example 2. We next consider the convergence rate of the numerical scheme (2.12) for (4.1) with $\varepsilon = 1$. The problem (4.1) is discretized using backward-Euler convolution quadrature and linear Galerkin finite element method. To investigate the convergence rate, we consider $I = 500$ independent realizations for each time step $\tau_k = 2^{-k}$, $k = 2, \dots, 6$. A surrogate for the exact solution $u(x, t_n, \omega_i)$,

$i = 1, 2, \dots, I$, is determined by using a much smaller time step $\tau = 2^{-13}$. In order to focus on the time discretization error, we solve the time-discrete stochastic PDE (2.12) using a sufficiently small spatial mesh size $h = 1/M = 2^{-9}$ so that the spatial discretization error is relatively negligible. Then the error $\mathbb{E}\|U^n - u(\cdot, t_n)\|^2$ for a fixed time step τ_k is approximated by

$$E(\tau_k) = \frac{1}{I} \sum_{i=1}^I \|U^n(\cdot, \omega_i) - u(\cdot, t_n, \omega_i)\|^2. \quad (4.2)$$

In [14], it is proved that the backward-Euler convolution quadrature for time-fractional PDE (1.5) is first-order convergent. Thus, by Theorem 2.1, the convergence order of the scheme (2.12) for problem (4.1) should be $O(\tau^{1-\alpha/2})$ in a one-dimensional spatial domain. Consequently, we expect the convergence rate

$$\frac{E(\tau_k)}{E(\tau_{k+1})} \approx \left(\frac{\tau_k}{\tau_{k+1}} \right)^{1-\alpha/2} = 2^{1-\alpha/2}$$

for successive halvings of the time step. We test the above result by taking $\alpha = 0.5$ for a subdiffusion setting and $\alpha = 1.3$ for a diffusion-wave setting. From (4.2), $\frac{E(\tau_k)}{E(\tau_{k+1})} \approx 1.682$ (order 0.75) for $\alpha = 0.5$ and $\frac{E(\tau_k)}{E(\tau_{k+1})} \approx 1.275$ (order 0.35) for $\alpha = 1.3$. Clearly, the results in Table 4.1 ($t_n = 1$) illustrate the sharp convergence rate.

Table 1: $\mathbb{E}\|U^n - u(\cdot, t_n)\|^2$ and convergence rates for Example 2

k	τ_k	$\alpha = 0.5$			$\alpha = 1.3$		
		$E(\tau_k)$	$\frac{E(\tau_k)}{E(\tau_{k+1})}$	order	$E(\tau_k)$	$\frac{E(\tau_k)}{E(\tau_{k+1})}$	order
2	2^{-2}	1.887e-3	—	—	7.638e-2	—	—
3	2^{-3}	1.302e-3	1.449	0.535	6.023e-2	1.268	0.343
4	2^{-4}	8.157e-4	1.600	0.678	4.591e-2	1.312	0.392
5	2^{-5}	5.032e-4	1.621	0.697	3.415e-2	1.344	0.427
6	2^{-6}	2.956e-4	1.702	0.767	2.475e-2	1.380	0.465

Example 3. Lastly, we consider the $\alpha = 1$ case that corresponds to the one-dimensional integer-order stochastic parabolic equation

$$\partial_t u(x, t) - \partial_x^2 u(x, t) = f(x, t) + \dot{W}(x, t) \quad (4.3)$$

for $0 \leq x \leq 1$, $0 < t \leq 1$, with homogenous Dirichlet boundary condition and 0 initial condition, where

$$f = 2tx^2(1-x)^2 - t^2(2-12x+12x^2).$$

The exact solution of the corresponding deterministic problem is $u_d(x, t) = t^2 x^2 (1-x)^2$.

We solve the stochastic equation (4.3) using the backward-Euler scheme

$$\frac{u_n - u_{n-1}}{\tau} - \partial_x^2 u_n = f^n + \frac{W(\cdot, t_n) - W(\cdot, t_{n-1})}{\tau}, \quad (4.4)$$

where spatial discretization is effected by the standard piecewise linear Galerkin finite element method. The mesh size is fixed at $h = 1/M = 2^{-9}$ so that spatial error is relatively negligible. Similarly, we chose $\tau_k = 2^{-k}$, $k = 2, \dots, 6$, and consider 500 independent realizations to investigate the convergence rate. A surrogate for the exact solution is determined by solving (4.4) with the linear finite element

method with $\tau = 2^{-13}$ and $h = 2^{-9}$. Then, the error $\mathbb{E}\|U^n - u(\cdot, t_n)\|^2$ is computed for each fixed time step τ_k . By Theorem 2.1, the convergence rate of scheme (4.4) is $O(\tau^{1/2})$ in the one-dimensional spatial setting. Thus, we expect the convergence rate

$$\frac{E(\tau_k)}{E(\tau_{k+1})} \approx \left(\frac{\tau_k}{\tau_{k+1}} \right)^{\frac{1}{2}} = 2^{1/2} \approx 1.414.$$

The temporal errors $\mathbb{E}\|U^n - u(\cdot, t_n)\|^2$ are presented in Table 4.2 at $t_n = 1$. Clearly, the numerical results are consistent with the theoretical analyses given in Theorem 2.1.

Table 2: $\mathbb{E}\|U^n - u(\cdot, t_n)\|^2$ and convergence rates for Example 3

k	τ_k	$E(\tau_k)$	$\frac{E(\tau_k)}{E(\tau_{k+1})}$	order
2	2^{-2}	2.252e-2	—	—
3	2^{-3}	1.566e-2	1.438	0.524
4	2^{-4}	1.095e-2	1.430	0.516
5	2^{-5}	7.545e-3	1.451	0.537
6	2^{-6}	5.138e-3	1.468	0.554

5 Conclusion

We considered the stability and convergence of numerical approximations of a stochastic time-fractional PDE by using the backward Euler convolution quadrature in time. By means of a discrete analogue of the inverse Laplace transform, we derived an integral representation of the numerical solution which was then used to prove the sharp convergence rate of the numerical approximation.

A The mild solution in $C([0, T]; L^2(\Omega; L^2(\mathcal{O})))$

In the cases $\alpha < \min(1, 2/d)$ and $\alpha > 1$ with $d = 1$, the mild solution of (1.6) (with space-time white noise) has been studied in different function spaces under different settings. For example, see [11, 16]. For the reader's convenience, in this appendix, we illustrate that the mild solution given by (2.5)-(2.6) is indeed well defined in $C([0, T]; L^2(\Omega; L^2(\mathcal{O})))$, a result used for the numerical analysis in this paper.

In (2.6), the formula (2.3) implies

$$\int_0^t E(t-s)\phi_j dW_j(s) = \phi_j \int_0^t h_j(t-s) dW_j(s) \quad (\text{A.1})$$

for a deterministic time-independent function $\phi_j \in L^2(\mathcal{O})$ and with the deterministic space-independent function $h_j(\cdot)$ given by

$$h_j(t-s) = \frac{1}{2\pi i} \int_{\Gamma_{\theta, \kappa}} e^{z(t-s)} z^{\alpha-1} (z^\alpha + \lambda_j)^{-1} dz.$$

By the theory of the Ito integral and the identity (A.1), each term in (2.6) is well defined in $C([0, T]; L^2(\Omega; L^2(\mathcal{O})))$. Because the one-dimensional Wiener processes $W_j(s)$, $j = 1, 2, \dots$, are independent of each other, it follows that

$$\sup_{t \in [0, T]} \mathbb{E} \left\| \sum_{j=\ell}^{\ell+m} \int_0^t E(t-s)\phi_j dW_j(s) \right\|^2$$

$$\begin{aligned}
&= \sup_{t \in [0, T]} \sum_{j=\ell}^{\ell+m} \int_0^t \|E(t-s)\phi_j\|^2 ds = \sup_{t \in [0, T]} \sum_{j=\ell}^{\ell+m} \int_0^t \|E(s)\phi_j\|^2 ds \\
&\leq \sum_{j=\ell}^{\ell+m} \int_0^T \|E(s)\phi_j\|^2 ds \\
&= \int_0^T \sum_{j=\ell}^{\ell+m} \left| \frac{1}{2\pi i} \int_{\Gamma_{\theta, \kappa}} e^{zs} z^{\alpha-1} (z^\alpha + \lambda_j)^{-1} dz \right|^2 ds \\
&\leq C \int_0^T \sum_{j=\ell}^{\ell+m} \left(\int_{\Gamma_{\theta, \kappa}} \frac{1}{|z|^\beta} |dz| \right) \left(\int_{\Gamma_{\theta, \kappa}} \left| \frac{z^\alpha}{z^\alpha + \lambda_j} \right|^2 \frac{|e^{2zs}|}{|z|^{2-\beta}} |dz| \right) ds \\
&\quad + C \int_0^T \sum_{j=\ell}^{\ell+m} \left(\int_{\Gamma_{\theta, \kappa}^\kappa} \frac{1}{|z|^\beta} |dz| \right) \left(\int_{\Gamma_{\theta, \kappa}^\kappa} \left| \frac{z^\alpha}{z^\alpha + \lambda_j} \right|^2 \frac{|e^{2zs}|}{|z|^{2-\beta}} |dz| \right) ds \\
&\leq C \int_0^T \sum_{j=\ell}^{\ell+m} \left(\int_\kappa^\infty \frac{1}{r^\beta} dr \right) \left(\int_\kappa^\infty \left| \frac{r^\alpha}{r^\alpha + \lambda_j} \right|^2 \frac{e^{-2rs|\cos(\theta)|}}{r^{2-\beta}} dr \right) ds \quad (\text{use Lemma 3.2}) \\
&\quad + C \int_0^T \sum_{j=\ell}^{\ell+m} \left(\int_{-\theta}^\theta \frac{1}{\kappa^\beta} \kappa d\varphi \right) \left(\int_{-\theta}^\theta \left| \frac{\kappa^\alpha}{\kappa^\alpha + \lambda_j} \right|^2 \frac{e^{2\kappa s \cos(\varphi)}}{\kappa^{2-\beta}} \kappa d\varphi \right) ds \\
&\leq C \kappa^{1-\beta} \int_0^T \int_\kappa^\infty \sum_{j=\ell}^{\ell+m} \left| \frac{r^\alpha}{r^\alpha + \lambda_j} \right|^2 \frac{e^{-2rs|\cos(\theta)|}}{r^{2-\beta}} dr ds \\
&\quad + C \kappa^{1-\beta} \int_0^T \int_{-\theta}^\theta \sum_{j=\ell}^{\ell+m} \left| \frac{\kappa^\alpha}{\kappa^\alpha + \lambda_j} \right|^2 \frac{e^{2\kappa s \cos(\varphi)}}{\kappa^{2-\beta}} \kappa d\varphi ds,
\end{aligned}$$

where $\beta \in (1, 2 - \alpha d/2)$.

In view of Lemma 3.2, $\sum_{j=1}^\infty \left| \frac{r^\alpha}{r^\alpha + \lambda_j} \right|^2 \leq C r^{\alpha d/2}$ implies $\sum_{j=\ell}^\infty \left| \frac{r^\alpha}{r^\alpha + \lambda_j} \right|^2 \rightarrow 0$ as $\ell \rightarrow \infty$ and

$$\begin{aligned}
\int_0^T \int_\kappa^\infty \sum_{j=\ell}^\infty \left| \frac{r^\alpha}{r^\alpha + \lambda_j} \right|^2 \frac{e^{-2rs|\cos(\theta)|}}{r^{2-\beta}} dr ds &\leq \int_0^T \int_\kappa^\infty \sum_{j=1}^\infty \left| \frac{r^\alpha}{r^\alpha + \lambda_j} \right|^2 \frac{e^{-2rs|\cos(\theta)|}}{r^{2-\beta}} dr ds \\
&\leq C \int_0^T \int_\kappa^\infty r^{\alpha d/2} \frac{e^{-2rs|\cos(\theta)|}}{r^{2-\beta}} dr ds \\
&\leq C \int_\kappa^\infty r^{\alpha d/2 + \beta - 3} (1 - e^{-2rT|\cos(\theta)|}) dr \\
&\leq C \kappa^{\alpha d/2 + \beta - 2}.
\end{aligned}$$

The Lebesgue dominated convergence theorem implies that

$$\lim_{\ell \rightarrow \infty} \kappa^{1-\beta} \int_0^T \int_\kappa^\infty \sum_{j=\ell}^\infty \left| \frac{r^\alpha}{r^\alpha + \lambda_j} \right|^2 \frac{e^{-2rs|\cos(\theta)|}}{r^{2-\beta}} dr ds = 0.$$

Similarly, $\sum_{j=\ell}^\infty \left| \frac{\kappa^\alpha}{\kappa^\alpha + \lambda_j} \right|^2 \leq C \kappa^{\alpha d/2}$ implies $\sum_{j=\ell}^\infty \left| \frac{\kappa^\alpha}{\kappa^\alpha + \lambda_j} \right|^2 \rightarrow 0$ as $\ell \rightarrow \infty$, and

$$\int_0^T \int_{-\theta}^\theta \sum_{j=\ell}^\infty \left| \frac{\kappa^\alpha}{\kappa^\alpha + \lambda_j} \right|^2 \frac{e^{2\kappa s \cos(\varphi)}}{\kappa^{2-\beta}} \kappa d\varphi ds \leq \int_0^T \int_{-\theta}^\theta \sum_{j=1}^\infty \left| \frac{\kappa^\alpha}{\kappa^\alpha + \lambda_j} \right|^2 \frac{e^{2\kappa s \cos(\varphi)}}{\kappa^{2-\beta}} \kappa d\varphi ds$$

$$\begin{aligned}
&\leq C \int_0^T \int_{-\theta}^{\theta} \kappa^{\alpha d/2} \frac{e^{2\kappa s}}{\kappa^{2-\beta}} \kappa d\varphi ds \\
&\leq C \int_{-\theta}^{\theta} \kappa^{\alpha d/2+\beta-2} (e^{2\kappa T-1}) d\varphi \\
&\leq C \kappa^{\alpha d/2+\beta-2}.
\end{aligned}$$

Again, the Lebesgue dominated convergence theorem implies that

$$\lim_{\ell \rightarrow \infty} \kappa^{1-\beta} \int_0^T \int_{-\theta}^{\theta} \sum_{j=\ell}^{\infty} \left| \frac{\kappa^\alpha}{\kappa^\alpha + \lambda_j} \right|^2 \frac{e^{2\kappa s \cos(\varphi)}}{\kappa^{2-\beta}} \kappa d\varphi ds = 0.$$

Overall, we have

$$\sup_{t \in [0, T]} \mathbb{E} \left\| \sum_{j=\ell}^{\ell+m} \int_0^t E(t-s) \phi_j dW_j(s) \right\|^2 \rightarrow 0 \quad \text{as } \ell \rightarrow \infty$$

which implies that the sequence

$$\sum_{j=1}^{\ell} \int_0^t E(t-s) \phi_j dW_j(s), \quad \ell = 1, 2, \dots$$

is a Cauchy sequence in $C([0, T]; L^2(\Omega; L^2(\mathcal{O})))$. Consequently, the sequence converges to a function $u \in C([0, T]; L^2(\Omega; L^2(\mathcal{O})))$, which is the mild solution defined by (2.6).

Let L_2^0 denote the space of Hilbert-Schmidt operators on $L^2(\mathcal{O})$ (cf. [5, Appendix C]) with the operator norm

$$\|E(t-s)\|_{L_2^0} = \left(\sum_{j=1}^{\infty} \|E(t-s) \phi_j\|_{L^2(\mathcal{O})}^2 \right)^{\frac{1}{2}}. \quad (\text{A.2})$$

The above analysis clearly shows that

$$\int_0^t \|E(t-s)\|_{L_2^0}^2 ds < \infty. \quad (\text{A.3})$$

In view of [5, Proposition 4.20 and page 99], the stochastic integral (2.5) is well defined in $L^2(\Omega; L^2(\mathcal{O}))$, and (2.5) coincides with the series representation (2.6) ([5, section 4.2.2]).

Acknowledgements. The authors would like to thank Professor Xiaobing Feng for many valuable suggestions and comments.

Funding. The research of M. Gunzburger and J. Wang was supported in part by the USA National Science Foundation grant DMS-1315259 and by the USA Air Force Office of Scientific Research grant FA9550-15-1-0001. The research of B. Li was supported by a grant (A/C code: 1-ZE6L) from The Hong Kong Polytechnic University.

References

- [1] E. J. Allen, S. J. Novosel, and Z. Zhang. Finite element and difference approximation of some linear stochastic partial differential equations. *Stochastics and Stochastics Reports*, 64, 1998.
- [2] M. P. Calvo, E. Cuesta, and C. Palencia. Runge-Kutta convolution quadrature methods for well-posed equations with memory. *Numer. Math.*, 107(4):589–614, 2007.

- [3] P. Clément and G. Da Prato. Some results on stochastic convolutions arising in Volterra equations perturbed by noise. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 7(3):147–153, 1996.
- [4] E. Cuesta, C. Lubich, and C. Palencia. Convolution quadrature time discretization of fractional diffusion-wave equations. *Math. Comp.*, 75(254):673–696, 2006.
- [5] G. Da Prato and J. Zabczyk. *Stochastic Equations in Infinite Dimensions*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, second edition, 2014.
- [6] Q. Du and T. Zhang. Numerical approximation of some linear stochastic partial differential equations driven by special additive noises. *SIAM J. Numer. Anal.*, 40(4):1421–1445, 2002.
- [7] I. Gyöngy and D. Nualart. Implicit scheme for stochastic parabolic partial differential equations driven by space-time white noise. *Potential Analysis*, 7(4):725–757, 1997.
- [8] B. Jin, R. Lazarov, and Z. Zhou. Two fully discrete schemes for fractional diffusion and diffusion-wave equations with nonsmooth data. *SIAM J. Sci. Comput.*, 38(1):A146–A170, 2016.
- [9] B. Jin, B. Li, and Z. Zhou. An Analysis of the Crank-Nicolson Method for Subdiffusion. . *IMA J. Numer. Anal.*, 2016, DOI: 10.1093/imanum/drx019.
- [10] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. *Theory and Application of Fractional Differential Equations*. Elsevier Science B.V., Amsterdam, 2006.
- [11] M. Kovács and J. Printems. Strong order of convergence of a fully discrete approximation of linear stochastic Volterra type evolution equation. *Math. Comp.*, 83(289):2325–2346, 2014.
- [12] A. Laptev. Dirichlet and Neumann eigenvalue problems on domains in Euclidean spaces. *J. Functional Analysis*, 151:531–545, 1997.
- [13] P. Li and S. Yau. On the Schrödinger equation and the eigenvalue problem. *Commun. Math. Phys.*, 88:309–318, 1983.
- [14] C. Lubich, I. H. Sloan, and V. Thomée. Nonsmooth data error estimates for approximations of an evolution equation with a positive-type memory term. *Math. Comp.*, 65(213):1–17, 1996.
- [15] W. McLean and K. Mustapha. Time-stepping error bounds for fractional diffusion problems with non-smooth initial data. *J. Comput. Phys.*, 293:201–217, 2015.
- [16] J. B. Mijena and E. Nane. Space-time fractional stochastic partial differential equations. *Stochastic Process. Appl.*, 125:3301–3326, 2015.
- [17] K. Mustapha and D. Schötzau. Well-posedness of hp -version discontinuous Galerkin methods for fractional diffusion wave equations. *IMA J. Numer. Anal.*, 34(4):1426–1446, 2014.
- [18] Y. Yan. Galerkin finite element methods for stochastic parabolic partial differential equations. *SIAM J. Numer. Anal.*, 43(4):1363–1384, 2005.